Int. J. Solids Structures Vol. 24, No. 6, pp. 545–556, 1988
Printed in Great Britain.
0020-7683/88 \$3.00 + .00

ON THE COMPLEMENTARY BOUNDING THEOREMS FOR LIMIT ANALYSIS

YANG GAO
Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, U.S.A.

(Received 4 February 1987; in revised form 20 August 1987)

Abstract—A unified theory for the complementary-dual bounding theorems of limit analysis is established with the aid of convex analysis. Based on the property of superpotential in this theory, various variational principles are constructed, which include a new lower bound theorem, a more precise estimation for the safety factor and a penalty–duality type variational principle. Furthermore, an efficient penalty–duality algorithm for limit analysis is suggested and the applications of these theorems are illustrated.

1. INTRODUCTION

It is well known that the classical lower bound theorem of limit analysis is actually equivalent to a non-linear optimization problem with both equality (equilibrium condition) and inequality constraints (plastic yield condition). In engineering problems, it is rather difficult to choose proper statically admissible functions. In order to relax the yield condition in this problem, various methods have been suggested by different authors[1–5]. The duality about this variational problem has been discussed by Temam and Strang[6, 7] based on the theory of convex analysis.

It has been pointed out in Ref. [8] that in the variational boundary value problems of a mathematical and physical system, any condition concerning a physical property (such as incompressible condition, plastic yield condition and friction boundary condition, etc.) cannot be considered as a variational constraint, this kind of condition should appear in the variational functional by introducing a so-called superpotential[9, 10], the subdifferential of this superpotential will yield the corresponding physical law. Hence, a universal complementary-duality principle about the variational boundary value problem has been established in Ref. [8], which shows that there exists an elegant symmetry in the mathematical and physical system.

According to this universal principle, a true complementary bound theorem has been established in this paper by using Fenchel transformation. In this bounding theorem, the plastic yield condition is relaxed by the complementary plastic superpotential, its subdifferential yields the plastic flow constitutive equation. Based on the property of this superpotential, various variational principles about limit analysis are established, and a new lower bound theorem is proved. An efficient penalty-duality algorithm is suggested to solve limit analysis problems. Applications are illustrated by examples of structure analysis.

2. SUPERPOTENTIAL AND GOVERNING EQUATIONS

Let Ω be an open, bounded, connected subset of R^3 with a Lipschitz boundary Γ, E and Σ the admissible strain and stress space, respectively

$$E := \{\varepsilon \in L^p(\Omega) \mid \varepsilon = \{\varepsilon_{ij}\} = \varepsilon^t, \quad i,j = 1,2,3\}$$

$$\Sigma := \{\sigma \in L^q(\Omega) \mid \sigma = \{\sigma_{ij}\} = \sigma^t, \quad i,j = 1,2,3\}$$

where p, q are dual numbers: $1/p + 1/q = 1$. The bilinear form $\langle *, * \rangle : E \times \Sigma \to R$ is defined as $\langle \sigma, \varepsilon \rangle = \sigma_{ij}\varepsilon_{ij}$. For a rigid-perfectly plastic material (Levy–Mises media), the constitutive law may be written as

$$\left.\begin{array}{ll} \operatorname{tr} \varepsilon = 0 & \text{a.e. in } \Omega \\ f(\sigma) \leqslant 0 & \text{a.e. in } \Omega \\ \varepsilon^d = \lambda \dfrac{\partial f(\sigma)}{\partial \sigma} \Phi(\phi) \quad \text{or} \quad \sigma^d = \dfrac{\sigma_b}{|\varepsilon^d|} \varepsilon^d \Phi(\psi) & \text{in } \Omega \end{array}\right\} \tag{1}$$

where "tr" is the trace operator, f the plastic yield function

$$f(\tau^d) = |\tau^d| - \sigma_b = \sqrt{(\tau_{ij}^d \tau_{ij}^d)} - \sigma_b \tag{2}$$

τ^d the stress deviator, σ_b a material constant, $\lambda \geqslant 0$ the plastic flow factor, and Φ the jump function

$$\Phi(\zeta) = \begin{cases} 1 & (\zeta \geqslant 0) \\ 0 & (\zeta < 0) \end{cases} \tag{3}$$

$\phi(x) : \Omega \to R$ is the stress type dividing-domain function[8, 11], $\phi(x) = f(\tau(x))$; $\psi(x) : \Omega \to R$ is the conjugate dividing-domain function, $\psi(x) = |\varepsilon^d(x)|$.

Let

$$C := \{\varepsilon \in E \mid \operatorname{tr} \varepsilon = 0 \quad \text{a.e. in } \Omega\} \tag{4}$$

$$K := \{\sigma \in \Sigma \mid f(\sigma) \leqslant 0 \quad \text{a.e. in } \Omega\}. \tag{5}$$

Then, the superpotential function for rigid-perfect plasticity may be written as

$$w(\varepsilon) = \sigma_b |\varepsilon^d| \Phi(\psi) + \Psi_C(\varepsilon) \tag{6}$$

where $\Psi_C : E \to (-\infty, \infty]$ is the indicator function of set C

$$\Psi_C(\varepsilon) = \begin{cases} 0 & \text{if } \varepsilon \in C \\ \infty & \text{if } \varepsilon \notin C. \end{cases} \tag{7}$$

It is obvious that $w : E \to (-\infty, \infty]$ is convex, lower semicontinuous, so constitutive law (1) may be written in the following unified form:

$$\tau \in \partial w(\varepsilon). \tag{8}$$

If C is non-empty, then one should have (cf. e.g. Ref. [12])

$$\partial w(\varepsilon) = \frac{\sigma_b}{|\varepsilon^d|} \varepsilon^d \Phi(\psi) + \partial \Psi_C(\varepsilon) \tag{9}$$

where

$$\partial \Psi_C(\varepsilon) = \begin{cases} \zeta \mathbf{I} & \text{if } \varepsilon \in C \\ \emptyset & \text{if } \varepsilon \notin C \end{cases} \tag{10}$$

ζ is an undetermined parameter, $\mathbf{I}$ a unit tensor.

The conjugate superpotential of $w(\varepsilon)$ may be given using the Fenchel transformation

$$w^*(\tau) := \sup_{\varepsilon \in E} \{\langle \tau, \varepsilon \rangle - w(\varepsilon)\}. \tag{11}$$

It should be noted that if $\varepsilon \in C$, then one has

$$w(\varepsilon) = \sigma_b|\varepsilon^d|\Phi(\psi) = \sup_{\tau\in K}\langle|\tau^d|, |\varepsilon^d|\Phi(\psi)\rangle = \Psi_K^*(\varepsilon). \tag{12}$$

In convex analysis, Ψ_K^* is the support function of convex set K, its conjugate function must be the indicator function of set K, hence one has

$$\begin{aligned} w^*(\tau) &= \sup_{\varepsilon\in E}\{\langle\tau,\varepsilon\rangle - w(\varepsilon)\} \\ &= \sup_{\varepsilon\in C}\{\langle\tau,\varepsilon\rangle - \Psi_K^*(\varepsilon)\} \\ &= \Psi_K(\tau). \end{aligned} \tag{13}$$

Therefore, the complementary superpotential for rigid–perfect plasticity is just the indicator function of convex set K. Thus, the inverse form of constitutive law (8) may be written as

$$\varepsilon\in\partial w^*(\tau) = \begin{cases} \lambda\dfrac{\partial f(\tau)}{\partial\tau} & \text{if } f(\tau) = 0,\ \lambda\geqslant 0 \\ \{0\} & \text{if } f(\tau) < 0 \\ \emptyset & \text{if } f(\tau) > 0. \end{cases} \tag{14}$$

It may be proved by convex analysis that the following conditions are equivalent:

(a) $\tau\in\partial w(\varepsilon)$;
(b) $\varepsilon\in\partial w^*(\tau)$;
(c) $w(\varepsilon)+w^*(\tau) = \langle\varepsilon,\tau\rangle$.

Assume that U is the space of admissible velocity, L the conjugate space of U, $D: U\to E$ the linear deformation operator, $Dv = \frac{1}{2}(\nabla v+v\nabla)$; $D^*: \Sigma\to L$ the conjugate operator, $D^*\tau = -\nabla\cdot\tau$, then, the boundary value problem of limit analysis becomes finding the safety factor $S_c > 0$ and field functions (u,σ) such that

$$\begin{aligned} Du-\varepsilon = 0 \quad &\text{in } \Omega, \qquad u = 0 \quad \text{on } \Gamma_u; \\ D^*\sigma - b = 0 \quad &\text{in } \Omega, \qquad \sigma\cdot n - S_c\cdot t = 0 \quad \text{on } \Gamma_t; \\ \sigma\in\partial w(\varepsilon) \quad &\text{or} \quad \varepsilon\in\partial w^*(\sigma) \quad \text{in } \Omega. \end{aligned} \tag{15}$$

It is useful in limit analysis if one puts

$$\int_{\Gamma_t} t\cdot u\,\mathrm{d}x = 1 \tag{16}$$

$$D^*\sigma = l(\sigma) = \begin{cases} -\nabla\cdot\sigma & \text{in } \Omega \\ n\cdot\sigma & \text{on } \Gamma. \end{cases} \tag{17}$$

3. COMPLEMENTARY BOUNDING THEOREMS

Define the kinematic admissible subspace U_a such that

$$U_a := \left\{v\in U\,|\,v = 0 \text{ on } \Gamma_u,\ \int_{\Gamma_t} v\cdot t\,\mathrm{d}x = 1\right\}. \tag{18}$$

The upper bound of the safety factor $S_u: U_a\to(-\infty,\infty]$ may be given as

$$S_u(v) = \int_\Omega [\sigma_b|\varepsilon^d(v)|\Phi(\psi)+\Psi_C(\varepsilon(v))]\,\mathrm{d}x - \int_\Omega v\cdot b\,\mathrm{d}x. \tag{19}$$

Theorem 1. *For all $v\in U_a$, the safety factor S_c is the infimum of $S_u(v)$*, i.e.

$$S_c = \inf_{v\in U_a} S_u(v). \tag{20}$$

Proof. An element u of U_a is the optimization solution of problem (20) if and only if

$$0\in\partial S_u(u). \tag{21}$$

If $w(Dv)$ is continuous at point u, then the subdifferential extremum condition (21) will yield the equilibrium-constitutive equation[8]

$$D^*\partial w(Du)\ni D^*\sigma(Du) = \begin{cases} b & \text{in } \Omega \\ S_c\cdot t & \text{on } \Gamma_t \end{cases} \tag{22}$$

i.e. the stress associated with solution u is statically admissible. So the optimization solution u must be the complete solution of boundary value problem (15), and S_u gives rise to the safety factor. QED

If $v\in U_a\cap C$, then functional (19) becomes $S^+ : U_a\cap C\to(-\infty,\infty]$

$$S^+(v) = \int_\Omega \sigma_b|\varepsilon^d(v)|\,\mathrm{d}x - \int_\Omega b\cdot v\,\mathrm{d}x \tag{23}$$

one has the classical upper bound theorem

$$S_c = \inf S^+(v)\quad \forall v\in U_a\cap C. \tag{24}$$

In order to establish the dual theorem of Theorem 1, functional (19) can be written in the following form:

$$\Pi(\varepsilon(v),v) := W(\varepsilon(v)) - \int_\Omega b\cdot v\,\mathrm{d}x + \Psi_{U_a}(v) \tag{25}$$

where

$$W(\varepsilon) := \begin{cases} \int_\Omega w(\varepsilon)\,\mathrm{d}x & \text{if } w\in L^1(\Omega) \\ \infty & \text{otherwise.} \end{cases}$$

It is obvious that $\Pi : E\times U\to(-\infty,\infty]$ is a convex, lower semicontinuous, proper functional, so problem (21) may be written as

$$\Pi(Du,u) = \inf_{v\in U}\Pi(Dv,v). \tag{26}$$

According to the theory of convex analysis, the conjugate functional $\Pi^* : \Sigma\times L\to(-\infty,\infty]$ of $\Pi(\varepsilon,v)$ may be obtained by the Fenchel transformation

$$\Pi^*(\tau,-D^*\tau) := \sup_{\varepsilon\in E}\sup_{v\in U}\{\langle\tau,\varepsilon\rangle_\Omega + \langle -D^*\tau, v\rangle_\Omega - \Pi(\varepsilon,v)\} \tag{27}$$

where

$$\langle -l, v\rangle_\Omega = \int_\Omega \langle -l, v\rangle \, dx + \int_\Gamma \langle -l, v\rangle \, dx.$$

Considering eqns (13), one has

$$W^*(\tau) = \sup_{\varepsilon \in E} \{\langle \tau, \varepsilon\rangle_\Omega - W(\varepsilon)\} = \int_\Omega \Psi_K(\tau) \, dx. \tag{28}$$

Moreover, it is easy to prove that

$$\sup_{v \in U} \{\langle -D^*\tau, v\rangle_\Omega - \Psi_{U_a}(v) + \langle b, v\rangle_\Omega\} = \Psi_{\Sigma_a}(\tau) - S^-(\tau) \tag{29}$$

where Σ_a is the statically admissible space

$$\Sigma_a := \{\tau \in \Sigma \mid D^*\tau - b = 0 \text{ in } \Omega, \quad \tau \cdot n - S^- \cdot t = 0 \text{ on } \Gamma_t\} \tag{30}$$

and $S^- > 0$ is a statically admissible factor associated with $\tau \in \Sigma_a$. Thus one has the complementary variational functional of $\Pi(\varepsilon, v)$ (25)

$$\Pi^*(\tau, -D^*\tau) = W^*(\tau) - S^-(\tau) + \Psi_{\Sigma_a}(\tau). \tag{31}$$

Letting $S_l(\tau) : \Sigma_a \to [-\infty, \infty)$ be a concave, upper semicontinuous functional

$$S_l(\tau) := S^-(\tau) - \int_\Omega \Psi_K(\tau) \, dx \tag{32}$$

then the dual bound theorem of the limit analysis problem may be given as follows.

Theorem 2. *For all statically admissible fields* $\tau \in \Sigma_a$, *the safety factor* S_c *is the supremum of* $S_l(\tau)$, i.e.

$$S_c = \sup_{\tau \in \Sigma_a} S_l(\tau). \tag{33}$$

This is a real complementary lower bound theorem of limit analysis, it will be proved by the next theorem. If the statically admissible field satisfies the yield condition almost everywhere in Ω, then eqn (33) will degenerate into the classical lower bound theorem

$$S_c = \sup S^-(\tau) \quad \forall \tau \in \Sigma_a \cap K. \tag{34}$$

According to the theory of convex analysis[12], the Lagrangian $S_L : U \times \Sigma \to (-\infty, \infty]$ associated with functional (32) is[8]

$$S_L(v, \tau) := \frac{\int_\Omega [\tau \cdot Dv - \Psi_K(\tau) - b \cdot v] \, dx - \int_{\Gamma_u} v \cdot \tau \cdot n \, dx}{\int_{\Gamma_t} t \cdot v \, dx}. \tag{35}$$

It should be noted that S_L is a saddle functional, i.e.

$S_L(*, \tau) : U \to (-\infty, +\infty]$ is convex, lower semicontinuous;

$S_L(v, *) : \Sigma \to (-\infty, +\infty]$ is concave, upper semicontinuous.

So, the generalized variational principle of limit analysis may be given as follows.

Theorem 3. For any given admissible fields $(v, \tau) \in U \times \Sigma$, *the safety factor* S_c *is the saddle point value of the Lagrangian* $S_L(v, \tau)$, i.e.

$$S_c = \inf_{v \in U} \sup_{\tau \in \Sigma} S_L(v, \tau). \tag{36}$$

Proof. It is easy to prove that problem (36) is equivalent to the following problem: find (u, σ) such that for any given $(v, \tau) \in U \times \Sigma$

$$\mathbf{L}(u, \sigma) = \inf_{v \in U} \sup_{\tau \in \Sigma} \mathbf{L}(v, \tau) \tag{37}$$

where $\mathbf{L} : U \times \Sigma \to (-\infty, +\infty]$ is the Lagrangian associated with $\Pi^*(\tau, -D^*\tau)$:

$$\mathbf{L}(v, \tau) := \int_\Omega [\tau Dv - \Psi_K(\tau) - b \cdot v] \, \mathrm{d}x - \int_{\Gamma_u} v \cdot \tau \cdot n \, \mathrm{d}x - S_c \int_{\Gamma_t} t \cdot v \, \mathrm{d}x. \tag{38}$$

If dom $\Psi_K(\tau) \neq \emptyset$, then the extremum condition

$$(0, 0) \in \partial \mathbf{L}(u, \sigma) \tag{39}$$

will yield the Euler–Lagrange equations

$$\begin{aligned} Du \in \partial \Psi_K(\sigma), \quad D^*\sigma - b = 0 \quad \text{in } \Omega \\ u = 0 \text{ on } \Gamma_u, \quad \sigma \cdot n - S_c \cdot t = 0 \quad \text{on } \Gamma_t. \end{aligned} \tag{40}$$

Substituting eqns (40) into eqn (36), one must have

$$\inf_{v \in U} \sup_{\tau \in \Sigma} S_L(v, \tau) = S_L(u, \sigma) = S_c.$$

Hence, the proof is completed. QED

According to the complementary variational principle[8], it is easy to prove that

$$S_u(v) = \sup_{\tau \in \Sigma} S_L(v, \tau) \tag{41}$$

$$S_l(\tau) = \inf_{v \in U} S_L(v, \tau). \tag{42}$$

And one has *the complementary-dual extremum principle*: for any given $(v, \tau) \in U_a \times \Sigma_a$

$$S_l(\tau) \leqslant S_l(\sigma) = S_L(u, \sigma) = S_u(u) \leqslant S_u(v). \tag{43}$$

Theorems 2 and 3 are two important theorems, according to the property of superpotential, various variational principles for limit analysis may be constructed correctly.

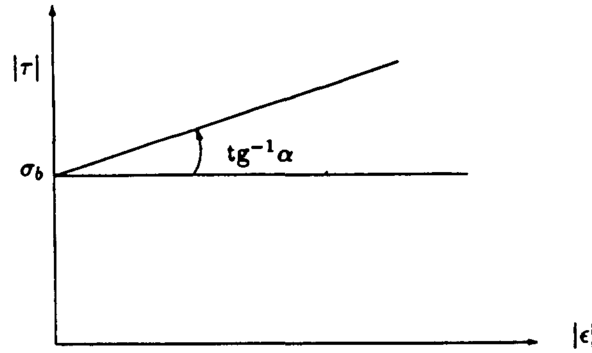

Fig. 1. Linear hardening rigid plastic model.

4. CONSTRUCTIONS OF VARIATIONAL PRINCIPLE

The simplest construction of the superpotential $\Psi_K(\tau)$ can be given by

$$\Psi_p(\tau,\alpha) := \frac{1}{2\alpha} f^2(\tau)\Phi(\phi) \tag{44}$$

where $\alpha > 0$ is a penalty factor, the jump function $\Phi(\phi)$ is given by eqn (3). It is obvious that for any given $\tau \in \Sigma$, one has

$$\Psi_K(\tau) = \sup_{\alpha>0} \Psi_p(\tau,\alpha) = \lim_{\alpha\to+0} \Psi_p(\tau,\alpha). \tag{45}$$

So the penalty-type variational principle of limit analysis may be established by substituting eqn (45) into eqn (33)

$$S_c = \lim_{\alpha\to+0} \sup_{\tau\in\Sigma_a} \left\{ S^-(\tau) - \int_\Omega \Psi_p(\tau,\alpha)\, \mathrm{d}x \right\}. \tag{46}$$

The physical significance of the penalty-type superpotential is also very clear, it comes from the plastic hardening media. Considering a rigid plastic material with linear hardening property shown in Fig. 1. For any given hardening parameter $\alpha > 0$, the plastic complementary energy is $w_\alpha^*(\tau) = \Psi_p(\tau,\alpha)$. When the hardening parameter $\alpha \to 0$, i.e. perfectly plastic media, the complementary energy $w_\alpha^*(\tau) \to w^*(\tau) = \Psi_K(\tau)$. Actually, the complementary energy variational principle for rigid plasticity was first obtained in 1983 in this way[13]. Based on the penalty-type variational principle (46), the penalty finite element model of limit analysis may be established, which accords a method of sequence optimization for boundary value problem (15). However, unfortunately, from the numerical analysis point of view, the discretized equations obtained from eqn (46) are quite often ill-conditioned if the penalty factors used are large enough, and the convergent rate of this method is rather slow. These disadvantages are inherent in the penalty function methods.

The complementary construction of penalty type is the duality-type construction

$$\Psi_d(\tau,\lambda) := \lambda f(\tau)\Phi(\phi) \tag{47}$$

where $\lambda \geqslant 0$ is the dual variable of function $f(\tau)$. For any given $\tau \in \Sigma$, one should have

$$\Psi_K(\tau) = \sup_{\lambda\geqslant 0} \Psi_d(\tau,\lambda). \tag{48}$$

Substituting eqn (48) into eqn (35), the duality-type generalized variational principle for safety factor S_c may be obtained as

$$S_c = \inf_{v \in U} \sup_{\tau \in \Sigma} \inf_{\lambda \geqslant 0} S_d(v, \tau, \lambda) \tag{49}$$

where

$$S_d(v, \tau, \lambda) := \frac{\displaystyle\int_\Omega [\tau D v - \lambda f(\tau)\Phi(\phi) - b \cdot v] \, \mathrm{d}x - \int_{\Gamma_u} v \cdot \tau \cdot n \, \mathrm{d}x}{\displaystyle\int_{\Gamma_t} t \cdot v \, \mathrm{d}x}. \tag{50}$$

This principle was first obtained in 1983[11] using the classical Lagrange multiplier method combined with a new variational technique of movable domain. It has been proved that for Mises' yield function (4), the dual variable (i.e. Lagrange multiplier) may be written as

$$\lambda(v) = |\varepsilon^d(v)|. \tag{51}$$

From the convex analysis point of view, it is difficult to solve the non-linear programming problem with inequality constraint using the classical Lagrange multiplier method. But, based on this generalized variational principle (49), two important theorems may be established.

Theorem 4. *For any given independent variables* $(v, \tau) \in U_a \times \Sigma_a$, *the following inequalities are true:*

$$S_m(v, \tau) \geqslant S^{-}(\tau) \quad \forall \tau \in \Sigma_a \cap K \tag{52}$$

$$S_m(v, \tau) \leqslant S^{+}(v) \quad \forall v \in U_a \cap C \tag{53}$$

where $S_m: U_a \times \Sigma_a \to (-\infty, +\infty]$ *is determined by putting* $\Phi(\phi) = 1$ *in functional* S_d

$$S_m(v, \tau) := \int_\Omega [\tau D v - \lambda(v) f(\tau)] \, \mathrm{d}x - \int_\Omega b \cdot v \, \mathrm{d}x. \tag{54}$$

Proof. Since $f(\tau) \leqslant 0$, $\lambda(v) \geqslant 0$ for any given $\tau \in K$ and $v \in U_a$, it is true that

$$S_m(v, \tau) - S^{-}(\tau) = -\int_\Omega \lambda(v) f(\tau) \, \mathrm{d}x \geqslant 0.$$

Moreover, for any given $v \in U_a \cap C$, one has

$$S^{+}(v) - S_m(v, \tau) = \int_\Omega [|\tau^d| \, |\varepsilon^d(v)| - \tau^d \varepsilon^d(v)] \, \mathrm{d}x.$$

Hence, the second inequality (53) is also true taking the Cauchy–Schwartz inequality into account. QED

Letting the body force $b = 0$ (in Ω), then one has the following.

Theorem 5. *For any given statically admissible field* $\tau \in \Sigma_a$, *if* $|\tau^d|\Phi(\phi)$ *is nonzero everywhere in* Ω, *then one has*

$$S_c \geqslant \frac{\sigma_b S^-(\tau)}{\max_{x\in\Omega}\{|\tau^d(x)|\Phi(\phi)\}}. \tag{55}$$

Proof. According to the duality-type generalized variational principle (49), one has

$$S_c = \sup_{\tau\in\Sigma}\inf_{v\in U} S_d(v,\tau,\lambda(v)) = \sup_{\tau\in\Sigma_a}\left\{S^-(\tau)-\int_\Omega \lambda(u)f(\tau)\Phi(\phi)\,\mathrm{d}x\right\}$$

where u is the solution of problem (20). So for any given $\tau\in\Sigma_a$, one has

$$\begin{aligned} S_c &\geqslant S^-(\tau)-\int_\Omega \lambda(u)f(\tau)\Phi(\phi)\,\mathrm{d}x \\ &\geqslant S^-(\tau)-\max_{x\in\Omega}\{f(\tau(x))\Phi(\phi(x))\}\int_\Omega \lambda(u)\,\mathrm{d}x. \end{aligned} \tag{56}$$

Since

$$S_c = \inf_{v\in U_a}\int_\Omega \sigma_b\lambda(v)\,\mathrm{d}x = \int_\Omega \sigma_b\lambda(u)\,\mathrm{d}x \tag{57}$$

Theorem 5 is proved by substituting eqn (57) into eqn (56). QED

Introducing both penalty factor and dual variable, an interesting penalty–duality type construction of superpotential $\Psi_K(\tau)$ may be obtained

$$\Psi_{pd}(\tau,\lambda,\alpha) := \frac{\alpha}{2}\left\{\left[\lambda+\frac{1}{\alpha}f(\tau)\right]^2\Phi(\phi_\alpha)-\lambda^2\right\} \tag{58}$$

where

$$\phi_\alpha := \lambda+\frac{1}{\alpha}f(\tau)$$

is the so-called penalty–duality dividing domain function. It may be proved that for any given $\tau\in\Sigma$

$$\Psi_K(\tau) = \sup_{\lambda\geqslant 0}\sup_{\alpha>0}\Psi_{pd}(\tau,\lambda,\alpha). \tag{59}$$

According to Theorem 3, the penalty–duality type generalized variational principle of limit analysis may be given as follows.

Theorem 6. There exists an $\alpha_* > 0$, such that for any given $\alpha\in(0,\alpha_*]$, the safety factor S_c is the stationary value of the following variational problem:

$$S_c = \inf_{v\in U}\sup_{\tau\in\Sigma}\inf_{\lambda\geqslant 0} S_{pd}(v,\tau,\lambda,\alpha) \tag{60}$$

where

$$S_{pd}(v,\tau,\lambda,\alpha) := \frac{\int_\Omega [\tau Dv - \Psi_{pd}(\tau,\lambda,\alpha)]\,\mathrm{d}x - \int_\Omega b\cdot v\,\mathrm{d}x - \int_{\Gamma_u} n\cdot\tau\cdot v\,\mathrm{d}x}{\int_{\Gamma_t} t\cdot v\,\mathrm{d}x}. \tag{61}$$

The proof of this theorem may be found in Ref. [9]. Based on this theorem, an efficient algorithm for the safety factor may be suggested.

Given the penalty–duality parameter $\alpha_n > 0$, $\lambda_n \geqslant 0$, determine σ_n, u_n, S_n by

$$S_n = \inf_{v\in U}\sup_{\tau\in\Sigma} S_{pd}(v,\tau,\lambda_n,\alpha_n). \tag{62}$$

Then, modify the penalty–duality parameter by

$$\alpha_{n+1} = \begin{cases} \gamma\alpha_n & \text{if } |f(\sigma_n)| \geqslant \theta|f(\sigma_{n-1})| \\ \alpha_n & \text{otherwise} \end{cases} \tag{63}$$

$$\lambda_{n+1} = \left\{\lambda_n + \frac{1}{\alpha_n} f(\sigma_n)\right\}\Phi(\phi_\alpha) \quad \text{in } \Omega \tag{64}$$

where $\gamma\in[0.1, 0.25]$, $\theta\in[0.1, 0.5]$ are constants determined by numerical experiments. The convergent rate of this algorithm is controlled directly by the penalty factor α_n. The smaller α_n, the faster the convergent rate. But the disadvantages of the pure penalty function method will appear in this algorithm if a sufficiently small penalty factor is taken.

5. APPLICATIONS

For the simply supported circular plate subjected to a uniform distributed load, the domain Ω is

$$\Omega = \{r,\theta \mid 0 \leqslant r \leqslant 1, 0 \leqslant \theta \leqslant 2\pi\}.$$

Mises' yield function f for this problem is

$$f(m_{ij}) = \sqrt{(m_r^2 + m_\theta^2 - m_r m_\theta)} - 1.$$

Choosing the try functions of deflection $w\in U_a$, the generalized variational functional (35) may be written as

$$S_L(w,m) = \int_0^{2\pi}\int_0^1 [-m_r w_{,rr} - m_\theta w_{,r}/r - \Psi_K(m_r,m_\theta)]r\,\mathrm{d}r\,\mathrm{d}\theta. \tag{65}$$

First of all the approximate solutions of the safety factor S_c are found using Theorem 4 with different try functions.

(i) $m_r = 1-r^2$, $m_\theta = 1$, $w = B_1(1-r^2)$

Using the condition

$$\int_0^{2\pi}\int_0^1 w\cdot r\,\mathrm{d}r\,\mathrm{d}\theta = 1$$

the constant B_1 may be determined as $B_1 = 3/\pi$. Substituting try function (i) into

Table 1. Numerical results of penalty–duality algorithm

	α_n	λ_n	η_n	$\bar{f}(\eta_n)$	$\Psi_{pd}(\eta_n, \lambda_n, \alpha_n)$	S_n
$n = 0$	0.1000	0.0000	1.3240	0.2070	0.7176	7.2240
$n = 1$	0.0400	2.0700	1.0960	-0.094%	0.0690	6.5040

$$S_m(w, m) = \int_0^{2\pi} \int_0^1 [-m_r w_{,rr} - m_\theta w_{,r}/r - \lambda(w) f(m_r, m_\theta)] r \, dr \, d\theta \tag{66}$$

one should have $S_m = 6.528$. With the same try functions, the classical bound theorems will give the upper and lower bound values of limit load: $S^+ = 6.93$, $S^- = 6.0$. The correct solution of this problem is $S_c = 6.51$, the relative error of the approximate solution S_m is only $(S_m - S_c)/S_c = 0.3\%$.

(ii) $m_r = 1 - r$, $m_\theta = 1$, $w = B_2(1-r)$

In the same way, one has $S_m = 6.35$, $S^+ = 8.0$, $S^- = 6.0$. Now, one can find the limit load of this problem by using the penalty–duality variational principle (60).

Choosing the try functions.

(iii) $m_r = \eta(1-r)$, $m_\theta = \eta$, $w = \dfrac{2}{\pi}(1-r)$

Here $\eta > 0$ is a parameter. Substituting try functions (iii) into penalty–duality variational functional (61), one has

$$S_{pd}(w, \eta, \lambda, \alpha) = 6\eta - \Psi_{pd}(\eta, \lambda, \alpha) \tag{67}$$

$$\Psi_{pd}(\eta, \lambda, \alpha) = 2\pi \int_0^1 \left\{ \left[\lambda + \frac{1}{\alpha} \bar{f}(m(\eta)) \right]^2 \Phi(\phi_\alpha) - \lambda^2 \right\} r \, dr. \tag{68}$$

The dividing domain function ϕ_α may be determined approximately by

$$\phi_\alpha = \lambda + \frac{1}{\alpha} \bar{f}(m(\eta)) = \lambda + \frac{1}{\alpha} \int_0^1 f(m(\eta)) r \, dr. \tag{69}$$

Choosing the primal value $\alpha_0 = 0.1$, $\lambda_0 = 0$, $\gamma = 0.4$, $\theta = 0.25$, for given precision $\omega = 0.001$, the numerical results obtained using the penalty–duality algorithm are shown in Table 1.

The numerical experiment shows that the penalty–duality algorithm possesses the higher precision and faster convergence rate. Based on this algorithm, a computer program consisting of about 2000 Fortran statements is developed in Ref. [8], and several engineering problems can be calculated[14].

Acknowledgements—The author offers his most grateful thanks to his advisors Professor K. C. Hwang and Professor G. Strang for their kind guidance and financial support. This research was supported in part by Army Research Office grant DAAG 29-83-K-0025 to Professor G. Strang.

REFERENCES

1. T. Mura and S. L. Lee, Application of variational principle of limit analysis. *Q. Appl. Math.* **21**(3), 243–248 (1963).
2. Chien Lin-xi and Zhong Wan-xie, On the limit analysis and suggesting a generalized variational principle. *Acta Mech. Sinica* **4** (1963).
3. T. Belytschko and P. G. Hodge, Jr., Plane stress limit analysis by finite element. *J. Engng Mech. Div., ASCE* **96**, 931–944 (1970).
4. D. N. Hutala, Finite element limit analysis of two-dimensional plane structures. In *Limit Analysis Using Finite Elements, Proc. Winter Ann. Mtg.* ASME, New York (Dec. 1976).

5. R. Casciano and L. Cascini, A mixed formulation and mixed finite elements for limit analysis. *Int. J. Numer. Meth. Engng* **18**, 211–243 (1982).
6. R. Temam and G. Strang, Duality and relaxation in the variational problems of plasticity. *J. Mec.* **19**(3), 493–529 (1980).
7. G. Strang, A minimax problem in plasticity theory. In *Functional Analysis Methods in Numerical Analysis* (Edited by M. Z. Nashed), Springer Lecture Notes 701, pp. 292–308 (1979).
8. Gao Yang, On the complementary principles for elasto-plastic system and panpenalty finite element method. Ph.D. Thesis, Tsinghua University, Beijing, China (1985).
9. Gao Yang and Hwang Keh-chih, On the complementary energy variational principles for Hencky plasticity. In *Proc. Int. Conf. Nonlinear Mechanics* (Edited by Chien Wei-zang). Science Press, Shanghai (1985).
10. J. J. Moreau, La notion de sur-potentiel et les liaisons unilaterates en elastostatique. *C. R. Acad. Sci.* **267**, 954–957 (1968).
11. Gao Yang, Extended bounding theorems of limit analysis. *Appl. Math. Mech.* **4**(4), 571–584 (1983).
12. I. Ekeland and R. Teman, *Convex Analysis and Variational Problems*. North-Holland, Amsterdam (1976).
13. Gao Yang, On the unified theory of variational principles for rigid-perfect plasticity and penalty–duality mixed finite element model. *J. Heifei Polytechnical Univ.* **3** (1983).
14. Gao Yang, Panpenalty finite element programming for plastic limit analysis. *Comput. Struct.* **28**, 749–755 (1988).